ON THE POSSIBILITY OF A UNIFIED APPROACH TO THE ANALYSIS OF DIFFERENT CASES OF RADIATIVE HEAT TRANSFER BETWEEN BODIES

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Three cases of radiative heat transfer are considered: between surfaces, between a surface and a volume, and between volumes. It is shown that all three cases can be described by means of the notion of a generalized mutual surface.

There are three fundamentally different cases of radiative heat transfer between bodies: radiative heat transfer between surfaces, between a surface and a volume, and between volumes. The first case arises when both bodies are opaque, the second when one body is opaque and the other semitransparent, and the third when both bodies are semitransparent.

The considerations in the present paper, as well as the whole theory of radiative heat transfer between bodies, are based on the assumption that the processes take place in a gray medium.

Radiative heat transfer between bodies is quantitatively described by the mutual radiation shape factor H.

The mutual shape factor of surfaces i and k is

$$\overline{H}(i, k) = \frac{1}{\pi} \int_{F_i} \int_{F_k} \cos\vartheta_i \cos\vartheta_k x^{-2} \exp(-\overline{k} x)\, dF_i\, dF_k . \tag{1}$$

The mutual radiation shape factor is, as we shall see below, a fundamental quantity in all cases of radiative heat transfer. It is often called the generalized mutual surface.

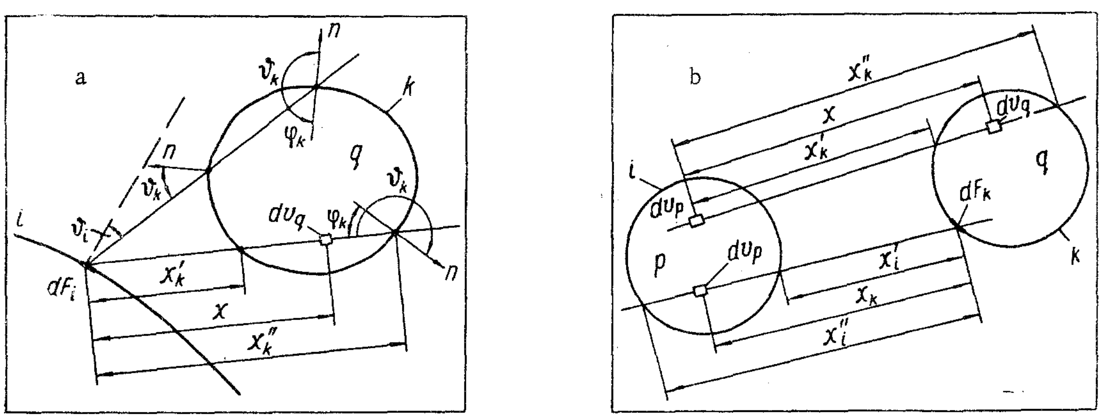

Fig. 1. Geometric quantities used in the determination of the mutual shape factor of a surface and a volume (a) and between two volumes (b).

The shape factor of surface i and volume q (Fig. 1a) is

$$H_{F-V}(i, q) = \frac{1}{\pi} \int_{F_i} \int_{V_q} k_q \cos\vartheta_i x^{-2} \exp(-\overline{k} x)\, dF_i\, dV_q , \tag{2}$$

and that of volumes p and q (Fig. 1b) is

$$H_{V-V}(p, q) = \frac{1}{\pi} \int_{V_p} \int_{V_q} k_p k_q x^{-2} \exp(-\overline{k} x)\, dV_p\, dV_q . \tag{3}$$

Using these factors, we obtain the expression for the energy emitted by surface i and intercepted by surface k

$$Q(i, k) = \varepsilon_i \sigma_0 T_i^4 \overline{H}(i, k). \tag{4}$$

Similarly, the energy emitted by surface i and absorbed by volume q is

$$Q_{\text{abs}}(i, q) = H_{F-V}(i, q)\, \varepsilon_i \sigma_0 T_i^4\,, \tag{5}$$

the energy emitted by volume q and intercepted by surface i is

$$Q(q, i) = H_{F-V}(i, q)\, \sigma_0 T_q^4, \tag{6}$$

and the energy emitted by volume p and absorbed by volume q is

$$Q_{\text{abs}}(p, q) = H_{V-V}(p, q)\, \sigma_0 T_p^4\,. \tag{7}$$

During the calculation of the quantity H(i, k), the integral in (1) should extend only over those surface elements which can "see" each other directly. For elements lying in the shade, the integrand is assumed to be zero. This method of calculating $\overline{H}(i, k)$ is suitable for cases involving heat transfer between surfaces.

In order to apply this notion to cases involving heat transfer between a surface and a volume, or between two volumes, it must be generalized by extending the integral also over the shaded parts of the surface. The expression for $\overline{H}(i, k)$ can be based either on the outer surface of the volume, or on the inner surface. If we adopt the convention that the angle ϑ is always measured in clockwise sense from a direction opposite to the ray to the normal to the surface element, then cos ϑ, as can be seen in Fig. 1, will be equal in magnitude for both methods of calculation, but it will be negative in the case of the first method, and positive in the second. Therefore the quantities $\overline{H}(i, k)$ calculated for a given part of the shaded surface by the two methods will also differ only in sign.

The integral of (1) over the whole outer surface (k) of a given volume shall be denoted by $\overline{H}_c(i_0, k_0)$.

Consider a beam emitted by a surface element dF_i (Fig. 1a). The quantity dV_q, representing the portion of the volume q pierced by this beam, can be replaced by $x^2 d\omega dx$. Equation (2) now becomes

$$H_{F-V}(i, q) = \frac{1}{\pi}\int_{F_i} dF_i \int_{\omega} \cos\vartheta_i\, d\omega \int_{x_k'}^{x_k''} k_q \exp(-\bar{k}x)\, dx. \tag{8}$$

We shall need the identity

$$d\exp(-\bar{k}x) = d\exp\Bigl(-\int_0^x kdx\Bigr) = -\exp(-\bar{k}x)\, k_q\, dx. \tag{9}$$

The quantity k_q in the last term of (9) is the value of k at the point q and appears upon differentiation of $\int_0^x kdx$. According to (9), we substitute $-d(\exp(-\bar{k}x))$ for $k_q(\exp(-\bar{k}x))$ in (8), and integrate from x_k' to x_k''. This yields

$$H_{F-V}(i, q) = \frac{1}{\pi}\int_{F_i} dF_i \int_{\omega} \cos\vartheta_i \exp(-\bar{k}x_k')\, d\omega - \frac{1}{\pi}\int_{F_i} dF_i \int_{\omega} \cos\vartheta_i \exp(-\bar{k}x_k'')\, d\omega. \tag{10}$$

The element of solid angle $d\omega$ can be written in terms of surface area elements of the portion of k which faces surface i

$$d\omega = (\cos\vartheta_k/(x_k')^2)\, dF_k, \tag{11}$$

as well as in terms of elements of the shaded surface, in which case

$$d\omega = (\cos\varphi_k/(x_k'')^2)\, dF_k = -(\cos\vartheta_k/(x_k'')^2)\, dF_k. \tag{12}$$

Substitute (11) into the first term of (10), and (12) into the second term. This yields

$$\begin{aligned} H_{F-V}(i, q) = \frac{1}{\pi}\int_{F_i} dF_i \int_{F_k'} \frac{\cos\vartheta_i \cos\vartheta_k}{(x_k')^2} \exp(-\bar{k}x_k')\, dF_k + \\ + \frac{1}{\pi}\int_{F_i} dF_i \int_{F_k''} \frac{\cos\vartheta_i \cos\vartheta_k}{(x_k'')^2} \exp(-\bar{k}x_k'')\, dF_k. \end{aligned} \tag{13}$$

In the first term the integration extends over the surface F_k', which faces the surface F_i, and in the second term it extends over the surface F_k'', which faces in the opposite direction. In both integrals the quantities x_k' and x_k'' represent the distances from surface elements dF_i to corresponding elements of F_k' and F_k''. Therefore the sum of the integrals in expression (13) is equal to the integral of the quantity $\frac{\cos\vartheta_i \cos\vartheta_k}{x_k^2}\exp(-\bar{k}x_k)$ over the whole surface F_k:

$$H_{F-V}(i, q) = \frac{1}{\pi}\int\limits_{F_i}\int\limits_{F_k} \cos\vartheta_i \cos\vartheta_k x_k^{-2} \exp(-\bar{k}x_k)\, dF_i\, dF_k\,. \tag{14}$$

Comparing this expression with formula (1), we see that the quantity $H_{F-V}(i, q)$ represents none other than the mutual shape factor of the surface i and the outer surface of the volume under consideration:

$$H_{F-V}(i, q) = \overline{H}_c(i_o, k_o). \tag{15}$$

Because a surface integeral is equal to the sum of the surface integrals over the parts of the surface,

$$\overline{H}_c(i_o, k_o) = \sum_\alpha\sum_\beta \overline{H}(\alpha_o, \beta_o). \tag{16}$$

The subscript "o" indicates that during the calculation of $\overline{H}$ the integration extends over the outer surface of the volume. In those cases in which the quantity $\overline{H}(\alpha, \beta)$ is obtained by integration over the inner surface, it should be substituted in (16) with a minus sign.

In the case when the absorption coefficient is equal to zero throughout the volume, we have

$$\overline{H}_c(i_o, k_o) = 0. \tag{17}$$

The elementary volume dV_q pierced by a beam emitted by the element dV_p (Fig. 1b) can be replaced by $x^2 d\omega dx$.

Carrying out this substitution, we reduce Eq. (3) to

$$H_{V-V}(p, q) = \frac{1}{\pi}\int\limits_{V_p} k_p\, dV_p \int\limits_\omega d\omega \int\limits_{x_k'}^{x_k''} k_q \exp(-\bar{k}x)\, dx. \tag{18}$$

In this case the integrand of the last integral can be replaced according to (9). After integration over x and replacement of $d\omega$ by (11) and (12), as we did in the case of radiative heat transfer between a surface and a volume, we obtain

$$H_{V-V}(p, q) = \frac{1}{\pi}\int\limits_{V_p} k_p dV_p \int\limits_{F_k} \cos\vartheta_k x_k^{-2} \exp(-\bar{k}x_k)\, dF_k. \tag{19}$$

Changing the order of integration in (19), we obtain

$$H_{V-V}(p, q) = \frac{1}{\pi}\int\limits_{F_k} dF_k \int\limits_{V_p} k_p \cos\vartheta_k x_k^{-2} \exp(-\bar{k}x_k)\, dV_p. \tag{20}$$

Fixing the position of the surface element dF_k, we replace dV_p by $x_k^2 d\omega dx$:

$$H_{V-V}(p, q) = \frac{1}{\pi}\int\limits_{F_k} dF_k \int\limits_\omega \cos\vartheta_k\, d\omega \int\limits_{x_i'}^{x_i''} k_p \exp(-\bar{k}x_k)\, dx_k. \tag{21}$$

The integrand in the last integral in (21) can be replaced according to the identity

$$d\exp(-\bar{k}x_k) = -\exp(-\bar{k}x_k)\, k_p\, dx, \tag{22}$$

which is analogous to (9). Formulas (11) and (12) are also valid, if the angle ϑ_k is replaced by ϑ_i. Let us replace the integrand in (21) in accordance with (22), and integrate the resulting expression over x_k between the boundaries of the volume.

After substituting (11) and (12) for $d\omega$, as we did before, we obtain

$$H_{V-V}(p, q) = \frac{1}{\pi} \int\limits_{F_i} \int\limits_{F_k} \cos\vartheta_i \cos\vartheta_k x_i^{-2} \exp(-\bar{k} x_i)\, dF_i\, dF_k . \tag{23}$$

Comparing this expression with (1), we find that the mutual shape factor of two volumes is none other than the mutual shape factor of their outer surfaces:

$$H_{V-V}(p, q) = \overline{H}_{c.c}(i_o, k_o). \tag{24}$$

The double subscript on the right-hand side of the equation indicates that during the calculation of $\overline{H}$ the integration extends over two closed surfaces.

By analogy with what we did before, we can write

$$\overline{H}_{c.c}(i_o, k_o) = \sum_{\alpha} \sum_{\beta} \overline{H}(\alpha_o, \beta_o). \tag{25}$$

Discussion of special cases. We shall find the mutual shape factor of two volumes p and q of cubic form, whose bases lie in a common plane, and with side faces perpendicular to the line joining the centers. The absorption coefficient is assumed to be constant.

According to formula (25), the mutual shape factor of the two volumes is equal to the algebraic sum of the 36 mutual shape factors of pairs of faces. However, in view of symmetry, many of the shape factors will be equal.

Let us choose the following seven shape factors as fundamental: 1) between the inner faces $\overline{H}(1, a)$; 2) between the outer faces $\overline{H}(2, b)$; 3) between an inner and an outer face $\overline{H}(1, c)$; 5) between an outer face and any of the side faces $\overline{H}(2, c)$; 6) between parallel side faces $\overline{H}(3, c)$; 7) between perpendicular side faces $\overline{H}(4, c)$.

Let us write down all 36 components $H(i_o, k_o)$, combining each face of volume p with each face of volume q. The subscript o shall remind us that the shape factors refer to the outer surfaces of the volumes.

For those components which refer to pairs of faces which see each other's outer sides, or those which see each other's inner sides, we drop the subscript o. For those components which refer to configurations in which one face sees an outer side and the other face sees an inner side, we drop the subscript o and add a minus sign.

Collecting all identical terms, and representing the result in terms of the seven shape factors listed above, we obtain

$$H_{V-V}(p, q) = \overline{H}_{c.c}(i_o, k_o) = \overline{H}(1, a) + \overline{H}(2, b) - 2\overline{H}(1, b) + 4\overline{H}(3, c) -$$
$$- 8\overline{H}(1, c) + 8\overline{H}(2, c) + 8\overline{H}(4, c).$$

Replacing the mutual shape factors by generalized irradiation factors ψ, we obtain

$$H_{V-V}(p, q) = F_i\, [\psi(1, a) + \psi(2, b) - 2\psi(1, b) +$$
$$+ 4\psi(3, c) - 8\psi(1, c) + 8\psi(2, c) + 8\psi(4, c)].$$

The method of calculation described here can also be used to calculate the mutual shape factor of volumes. Let us find the mutual shape factor for a simpler case: two volumes p and q each of which is bounded by two surfaces. One pair of these surfaces, one of each volume, see each other's inner sides, and the other pair see each other's outer sides (Fig. 2a). Using the present method, we can write immediately

$$H_{V-V}(p, q) = \overline{H}(1, 4) + \overline{H}(2, 3) - \overline{H}(2, 4) - \overline{H}(1, 3).$$

In the special case when the two volumes adjoin each other and are separated by a plane (Fig. 2b), the quantity $\overline{H}(1, 4)$ will be equal to the separating surface.

Representing the other terms by irradiation factors, we obtain

$$H_{V-V}(p, q) = F_1 + F_2 \psi(2, 3) - F_1\, [\psi(1,2) + \psi(1,3)].$$

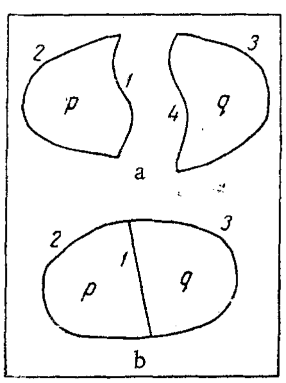

Fig. 2. Examples of two volumes with each bounding surface divided into two parts.

NOTATION

k – coefficient of absorption of the medium, variable throughout the volume; $\bar{k} = \frac{1}{x}\int\limits_0^x k\,dx$ – mean absorption coefficient on the ray x; k_p and k_q – true values of the absorption coefficient at the points p and q; ϑ_i and ϑ_k – angles between the direction of the beam and the normals to the surface elements dF_i and dF_k; ε_i – emissivity of surface i; T – absolute temperature; ω – solid angle subtended by volume q at the surface element dF_i; $\psi(i,k)$ – generalized irradiation factors for the irradiation of surface k by surface i; φ_k – angle between the direction opposite to the beam and the inner normal to a surface; α and β – parts of the surfaces i and k; F_i – the surface area of an individual face.

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